

# Compatible quadratic Poisson brackets related to a family of elliptic curves

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## Abstract

We construct nine pairwise compatible quadratic Poisson structures such that a generic linear combination of them is associated with an elliptic algebra in  $n$  generators. Explicit formulas for Casimir elements of this elliptic Poisson structure are obtained.

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# 1 Introduction

Two Poisson brackets  $\{\cdot, \cdot\}_0$  and  $\{\cdot, \cdot\}_1$  defined on the same finite dimensional vector space are said to be compatible if

$$\{\cdot, \cdot\}_u = \{\cdot, \cdot\}_0 + u\{\cdot, \cdot\}_1 \quad (1.1)$$

is a Poisson bracket for any constant  $u$ . Note that if  $\{\cdot, \cdot\}_{u_1, \dots, u_k} = \{\cdot, \cdot\}_0 + u_1\{\cdot, \cdot\}_1 + \dots + u_k\{\cdot, \cdot\}_k$  is a Poisson bracket for arbitrary  $u_1, \dots, u_k$ , then all brackets  $\{\cdot, \cdot\}_0, \dots, \{\cdot, \cdot\}_k$  are Poisson and pairwise compatible. Compatible Poisson structures play an important role in the theory of integrable systems [1, 2] and in differential geometry [3, 4]. A lot of examples of compatible Poisson structures are known [2]. Most of these are linear in certain coordinates. However, quadratic Poisson structures are also interesting. While the theory of linear Poisson structures is well-understood and possesses a classification theory<sup>1</sup>, the theory of quadratic Poisson algebras is more complicated. If the dimension of a linear space is larger than four, then no classification results for quadratic Poisson structures on this space are available. All known examples can be divided into two classes: rational and elliptic. In the elliptic case structure constants of a Poisson bracket are modular functions of a parameter  $\tau$ , a modular parameter of an elliptic curve. This elliptic curve appears naturally as a symplectic leaf of this elliptic Poisson structure [5].

Let  $Q_n(\tau, \eta)$  be an associative algebra defined by  $n$  generators  $\{x_i; i \in \mathbb{Z}/n\mathbb{Z}\}$  and quadratic relations [5]

$$\sum_{r \in \mathbb{Z}/n\mathbb{Z}} \frac{1}{\theta_{j-i-r}(-\eta, \tau)\theta_r(\eta, \tau)} x_{j-r} x_{i+r} = 0,$$

for all  $i \neq j \in \mathbb{Z}/n\mathbb{Z}$ . Here  $\theta_i(z, \tau)$  are  $\theta$ -functions with characteristics (see Appendix). It is known that for generic  $\eta$  the algebra  $Q_n(\tau, \eta)$  has the same size of graded components as the polynomial ring  $\mathbb{C}[x_1, \dots, x_n]$ . Moreover, if  $\eta = 0$ , then  $Q_n(\tau, \eta)$  is isomorphic to  $\mathbb{C}[x_1, \dots, x_n]$ . Therefore, for any fixed  $\tau$  we have a flat deformation of a polynomial ring. Let  $q_n(\tau)$  be the corresponding Poisson algebra. Symplectic leaves of this Poisson structure are known [5]. In particular, the center of this Poisson algebra is generated by one homogeneous polynomial of degree  $n$  if  $n$  is odd and by two homogeneous polynomials of degree  $\frac{n}{2}$  if  $n$  is even.

One can pose the following problems:

1. Do there exist Poisson structures compatible with the one in  $q_n(\tau)$ ?
2. Construct a maximal number of Poisson structures pairwise compatible and compatible with the one in  $q_n(\tau)$ .

It is easy to study these questions in the cases  $n = 3, 4$ .

Let  $n = 3$ . The Poisson bracket in  $q_3(\tau)$  can be written as

$$\{x_{\sigma_1}, x_{\sigma_2}\} = \frac{\partial P}{\partial x_{\sigma_3}}$$

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<sup>1</sup>The theory of linear Poisson structures coincides with the theory of Lie algebras.

where  $P$  is a certain homogeneous cubic polynomial in  $x_1, x_2, x_3$  and  $\sigma$  is an arbitrary even permutation. Moreover, this formula defines a Poisson bracket for an arbitrary polynomial  $P$  and all these brackets are pairwise compatible. In particular, there exist 10 linearly independent quadratic Poisson brackets because there are 10 linearly independent homogeneous cubic polynomials in 3 variables.

Let  $n = 4$ . The Poisson bracket in  $q_4(\tau)$  can be written as

$$\{x_{\sigma_1}, x_{\sigma_2}\} = \det \begin{pmatrix} \frac{\partial P}{\partial x_{\sigma_3}} & \frac{\partial P}{\partial x_{\sigma_4}} \\ \frac{\partial R}{\partial x_{\sigma_3}} & \frac{\partial R}{\partial x_{\sigma_4}} \end{pmatrix}$$

where  $P, R$  are certain homogeneous quadratic polynomials in  $x_1, \dots, x_4$  and  $\sigma$  is an arbitrary even permutation. Moreover, this formula defines a Poisson bracket for arbitrary polynomials  $P$  and  $R$ . If we fix  $R$  and vary  $P$ , we obtain an infinite family of pairwise compatible Poisson brackets. In particular, there exist 9 pairwise compatible quadratic Poisson brackets. Indeed, there are 10 quadratic polynomials in 4 variables and  $P$  should not be proportional to  $R$ .

If  $n > 4$ , then the situation is more complicated because the similar construction for  $q_n(\tau)$ ,  $n > 4$  does not exist. In this paper we construct nine pairwise compatible quadratic Poisson brackets<sup>2</sup> for arbitrary  $n$ . A generic linear combination of these Poisson brackets is isomorphic to  $q_n(\tau)$  where  $\tau$  depends on coefficients in this linear combination. Moreover, we think that this family of Poisson brackets is maximal. We have checked, that for  $n = 5, 6, \dots, 40$  there are no quadratic Poisson brackets that are compatible with all our nine Poisson brackets and are linearly independent of them. For these values of  $n$  there are no Poisson brackets compatible with all ours that are constant, linear, cubic and quartic.

Let us describe the contents of the paper. In section 2 we construct nine compatible quadratic Poisson structures on a certain  $n$ -dimensional linear space  $\mathcal{F}_n$ . This construction is slightly different for even and odd  $n$ . It is summarized in Remarks 1, 1' and 2, 2' as an algorithm for the computation of Poisson brackets between  $x_i$  and  $x_j$  for  $i, j = 0, 1, 2, 3, \dots, n$ . In section 3 we explain the functional version of the same construction. In sections 4 we describe symplectic leaves and Casimir elements of our Poisson algebras (see also [5, 6]). In the Conclusion we outline several open problems. In the Appendix we collect some notations and standard facts about elliptic and  $\theta$ -functions (see [8, 5] for details).

## 2 Algebraic construction of nine compatible quadratic Poisson brackets

### 2.1 Notations

Our construction of Poisson brackets is slightly different for even and odd  $n$ . We will use index *ev* (resp. *od*) for objects related to even (resp. odd)  $n$ .

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<sup>2</sup>Three of these were constructed in [6].

Let

$$P_{ev}(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4, \quad P_{od}(t) = a_0 + a_1t + a_2t^2 + a_3t^3, \quad Q(t) = b_0 + b_1t + b_2t^2 \quad (2.2)$$

be arbitrary polynomials of degree not larger than four, three and two correspondingly. Let  $\mathcal{F}_{ev}$  be a commutative associative algebra defined by generators  $f, g$  and the relation

$$g^2 = P_{ev}(f) + Q(f)g. \quad (2.3)$$

Let  $\mathcal{F}_{od}$  be a commutative associative algebra defined by generators  $f, g$  and the relation

$$(f + c)g^2 = P_{od}(f) + Q(f)g. \quad (2.4)$$

Here  $a_0, \dots, a_4, b_0, b_1, b_2, c$  are constants. Let  $D$  be a derivation of  $\mathcal{F}_{ev}$  and  $\mathcal{F}_{od}$  defined on its generators by

$$D(f) = 2g - Q(f), \quad D(g) = P'_{ev}(f) + Q'(f)g \quad (2.5)$$

for  $\mathcal{F}_{ev}$  and by

$$D(f) = 2(f + c)g - Q(f), \quad D(g) = P'_{od}(f) + Q'(f)g - g^2 \quad (2.6)$$

for  $\mathcal{F}_{od}$ .

Let  $\mathcal{F}$  be either  $\mathcal{F}_{ev}$  or  $\mathcal{F}_{od}$ . It is clear that  $\mathcal{F} \otimes \mathcal{F}$  is generated by  $f_1 = f \otimes 1, f_2 = 1 \otimes f, g_1 = g \otimes 1, g_2 = 1 \otimes g$  as an associative algebra. For an arbitrary element  $h \in \mathcal{F}$  we will use the notations  $h_1 = h \otimes 1, h_2 = 1 \otimes h$  for the corresponding elements in  $\mathcal{F} \otimes \mathcal{F}$ . Let  $\lambda_{ev} \in \text{Frac}(\mathcal{F}_{ev} \otimes \mathcal{F}_{ev})$  be an element of a field of fractions of  $\mathcal{F}_{ev} \otimes \mathcal{F}_{ev}$  defined by

$$(f_1 - f_2)\lambda_{ev} = g_1 + g_2 - \frac{1}{2}Q(f_1) - \frac{1}{2}Q(f_2), \quad (2.7)$$

or by

$$(g_1 - g_2)\lambda_{ev} = \frac{P_{ev}(f_1) - P_{ev}(f_2)}{f_1 - f_2} + \frac{Q(f_1) - Q(f_2)}{2(f_1 - f_2)}(g_1 + g_2).$$

These definitions are equivalent by virtue of (2.3). Let  $\lambda_{od} \in \text{Frac}(\mathcal{F}_{od} \otimes \mathcal{F}_{od})$  be an element of a field of fractions of  $\mathcal{F}_{od} \otimes \mathcal{F}_{od}$  defined by

$$(f_1 - f_2)\lambda_{od} = (f_1 + c)g_1 + (f_2 + c)g_2 - \frac{1}{2}Q(f_1) - \frac{1}{2}Q(f_2), \quad (2.8)$$

or by

$$(g_1 - g_2)\lambda_{od} = \frac{P_{od}(f_1) - P_{od}(f_2)}{f_1 - f_2} + \frac{Q(f_1) - Q(f_2)}{2(f_1 - f_2)}(g_1 + g_2) - g_1g_2.$$

These definitions are equivalent by virtue of (2.4). Note that  $\frac{H(f_1) - H(f_2)}{f_1 - f_2} \in S^2\mathcal{F} \subset \mathcal{F} \otimes \mathcal{F}$  for an arbitrary polynomial  $H$ . Indeed,  $\frac{f_1^m - f_2^m}{f_1 - f_2} = f_1^{m-1} + f_1^{m-2}f_2 + \dots + f_2^{m-1} \in S^2\mathcal{F}$ .

We define elements  $x_0, x_2, x_3, x_4, \dots \in \mathcal{F}$  by

$$x_{2i} = f^i, \quad x_{2i+3} = f^i g, \quad i = 0, 1, 2, \dots \quad (2.9)$$

Let  $\mathcal{F}_n \subset \mathcal{F}$  be an  $n$ -dimensional linear space with a basis  $\{x_0, x_2, x_3, \dots, x_n\} = \{x_0, x_i; 2 \leq i \leq n\}$ . Note that  $\mathcal{F}_n \subset \mathcal{F}$  is not a subalgebra of  $\mathcal{F}$ . We assume  $\mathcal{F}_n \subset \mathcal{F}_{ev}$  if  $n$  is even and  $\mathcal{F}_n \subset \mathcal{F}_{od}$  if  $n$  is odd. We will identify  $S^*\mathcal{F}_n$  with a polynomial algebra  $\mathbb{C}[x_0, x_2, \dots, x_n]$  in  $n$  variables. In particular:

$$f_1^i f_2^j + f_1^j f_2^i = x_{2i} x_{2j}, \quad f_1^i f_2^j g_2 + f_1^j f_2^i g_1 = x_{2i} x_{2j+3}, \quad (f_1^i f_2^j + f_1^j f_2^i)g_1 g_2 = x_{2i+3} x_{2j+3}. \quad (2.10)$$

## 2.2 Construction in the case of even $n$

**Proposition 1.** The following formula

$$\{\phi, \psi\} = n\lambda_{ev}(\phi_1\psi_2 - \psi_1\phi_2) + \phi_1D(\psi_2) + \phi_2D(\psi_1) - \psi_1D(\phi_2) - \psi_2D(\phi_1) \quad (2.11)$$

defines a quadratic Poisson bracket on the polynomial ring  $S^*\mathcal{F}_n = \mathbb{C}[x_0, x_2, \dots, x_n]$  where  $n$  is even. Here  $\phi, \psi \in \mathcal{F}_n$  and  $\{\phi, \psi\} \in S^2\mathcal{F}_n$ . This Poisson bracket is linear with respect to coefficients  $a_0, \dots, a_4, b_0, \dots, b_2$  of polynomials  $P_{ev}$ ,  $Q$  and, therefore, can be written in the form  $\{\cdot, \cdot\} = \{\cdot, \cdot\}_0 + \sum_{i=0}^4 a_i \{\cdot, \cdot\}_{i,1} + \sum_{j=0}^2 b_j \{\cdot, \cdot\}_{j,2}$  where  $\{\cdot, \cdot\}_0$ ,  $\{\cdot, \cdot\}_{i,1}$ ,  $\{\cdot, \cdot\}_{j,2}$  are pairwise compatible. Therefore, for each even  $n$  we have constructed nine compatible quadratic Poisson brackets in  $n$  variables.

**Proof.** The Jacobi identity is a consequence of a functional construction described in the next section. Let us check linearity with respect to coefficients of  $P_{ev}$ ,  $Q$ . Each of  $\phi, \psi \in \mathcal{F}_n \subset \mathcal{F}_{ev}$  can be of the form  $R(f)$  or  $R(f)g$  where  $R$  is a polynomial. Therefore, we have three cases:

**Case 1.** Let  $\phi = R(f)$ ,  $\psi = T(f)$ . We have

$$\begin{aligned} \{\phi, \psi\} &= \{R(f), T(f)\} \\ &= n\lambda_{ev}(R(f_1)T(f_2) - R(f_2)T(f_1)) + R(f_1)D(T(f_2)) + R(f_2)D(T(f_1)) \\ &\quad - T(f_1)D(R(f_2)) - T(f_2)D(R(f_1)) \\ &= n \frac{R(f_1)T(f_2) - R(f_2)T(f_1)}{f_1 - f_2} \left( g_1 + g_2 - \frac{1}{2}Q(f_1) - \frac{1}{2}Q(f_2) \right) + \\ &\quad (R(f_2)T'(f_1) - T(f_2)R'(f_1))(2g_1 - Q(f_1)) + (R(f_1)T'(f_2) - T(f_1)R'(f_2))(2g_2 - Q(f_2)) \end{aligned}$$

**Case 2.** Let  $\phi = R(f)$ ,  $\psi = T(f)g$ . We have

$$\begin{aligned} \{\phi, \psi\} &= \{R(f), T(f)\} \\ &= n\lambda_{ev}(R(f_1)T(f_2)g_2 - R(f_2)T(f_1)g_1) + R(f_1)D(T(f_2)g_2) + R(f_2)D(T(f_1)g_1) \\ &\quad - T(f_1)g_1D(R(f_2)) - T(f_2)g_2D(R(f_1)) \\ &= n \frac{R(f_1)T(f_2) - R(f_2)T(f_1)}{f_1 - f_2} g_1g_2 - \frac{n}{2} \frac{Q(f_1) - Q(f_2)}{f_1 - f_2} (R(f_1)T(f_2)g_2 + R(f_2)T(f_1)g_1) + \\ &\quad n \frac{R(f_1)T(f_2)P_{ev}(f_2) - R(f_2)T(f_1)P_{ev}(f_1)}{f_1 - f_2} \\ &\quad + R(f_1)T(f_2)(P'_{ev}(f_2) + Q'(f_2)g_2) + R(f_2)T(f_1)(P'_{ev}(f_1) + Q'(f_1)g_1) \\ &\quad + R'(f_1)T(f_2)(Q(f_1) - 2g_1)g_2 + R'(f_2)T(f_1)(Q(f_2) - 2g_2)g_1 \\ &\quad + R(f_1)T'(f_2)(2P_{ev}(f_2) + Q(f_2)g_2) + R(f_2)T'(f_1)(2P_{ev}(f_1) + Q(f_1)g_1). \end{aligned}$$

Here we used  $\lambda_{ev}(R(f_1)T(f_2)g_2 - R(f_2)T(f_1)g_1) =$

$$(g_1 + g_2) \frac{R(f_1)T(f_2) - R(f_2)T(f_1)}{2(f_1 - f_2)} \lambda_{ev}(f_1 - f_2) - \frac{1}{2}(R(f_1)T(f_2) + R(f_2)T(f_1))\lambda_{ev}(g_1 - g_2).$$

**Case 3.** Let  $\phi = R(f)g$ ,  $\psi = T(f)g$ . We have

$$\{\phi, \psi\} = \{R(f), T(f)\}$$

$$\begin{aligned}
&= n\lambda_{ev}((R(f_1)T(f_2) - R(f_2)T(f_1))g_1g_2) + R(f_1)g_1D(T(f_2)g_2) + R(f_2)g_2D(T(f_1)g_1) \\
&\quad - T(f_1)g_1D(R(f_2)g_2) - T(f_2)g_2D(R(f_1)g_1) \\
&= n\frac{R(f_1)T(f_2) - R(f_2)T(f_1)}{f_1 - f_2}(g_1^2g_2 + g_1g_2^2 - \frac{1}{2}Q(f_1)g_1g_2 - \frac{1}{2}Q(f_2)g_1g_2) \\
&\quad + (R(f_2)T'(f_1) - T(f_2)R'(f_1))(2g_1 - Q(f_1))g_1g_2 \\
&\quad + (R(f_1)T'(f_2) - T(f_1)R'(f_2))(2g_2 - Q(f_2))g_1g_2 \\
&\quad + (R(f_1)T(f_2) - R(f_2)T(f_1))(g_1D(g_2) - D(g_1)g_2) \\
&= n\frac{R(f_1)T(f_2) - R(f_2)T(f_1)}{f_1 - f_2}(P_{ev}(f_1)g_2 + g_1P_{ev}(f_2) + \frac{1}{2}Q(f_1)g_1g_2 + \frac{1}{2}Q(f_2)g_1g_2) \\
&\quad + (R(f_2)T'(f_1) - T(f_2)R'(f_1))(2P_{ev}(f_1) + Q(f_1)g_1)g_2 \\
&\quad + (R(f_1)T'(f_2) - T(f_1)R'(f_2))(2P_{ev}(f_2) + Q(f_2)g_2)g_1 \\
&\quad + (R(f_1)T(f_2) - R(f_2)T(f_1))(P'_{ev}(f_2)g_1 + Q'(f_2)g_1g_2 - P'_{ev}(f_1)g_2 - Q'(f_1)g_1g_2)
\end{aligned}$$

In these computations we use formulas (2.3) and (2.5) where  $f, g$  are replaced by  $f_1, g_1$  or  $f_2, g_2$ . Note that in each case we obtain an expression for  $\{\phi, \psi\}$  that is linear non-homogeneous in  $P_{ev}$ ,  $Q$  and bi-linear non-homogeneous in  $g_1, g_2$ . Using identifications (2.10) we can write each of these expressions as a quadratic homogeneous polynomial in  $x_0, x_2, x_3, \dots, x_n$  with coefficients linear in  $a_0, \dots, a_4, b_0, \dots, b_2$ .

**Remark 1.** Let us describe an algorithm for the computation of  $\{x_i, x_j\}$  as a quadratic polynomial in  $x_0, x_2, x_3, \dots, x_n$ . One uses the formulas for  $\{\phi, \psi\}$  from the proof of the proposition 1 where  $R(f) = f^i$ ,  $T(f) = f^j$  and  $P_{ev}$ ,  $Q$  are given by (2.2). For the computation of  $\{x_{2i}, x_{2j}\}$  the formula of case 1 is used, for the computation of  $\{x_{2i}, x_{2j+3}\}$  the formula of case 2 is used, and for the computation of  $\{x_{2i+3}, x_{2j+3}\}$  the formula of case 3 is used. These formulas give a polynomial in  $f_1, f_2, g_1, g_2$  linear in  $g_1, g_2$ . This polynomial is symmetric under transformations  $f_1 \leftrightarrow f_2$ ,  $g_1 \leftrightarrow g_2$ . By using identifications (2.10) the expressions can be written as a polynomial quadratic in  $x_0, x_2, x_3, \dots, x_n$ .

**Remark 2.** In all cases of the above remark  $\{x_i, x_j\}$  is linear non-homogeneous with respect to the eight coefficients  $a_0, \dots, b_2$  of polynomials  $P_{ev}$ ,  $Q$ . Therefore, nine compatible Poisson brackets can be obtained in the following way:

$$\{x_i, x_j\}_0 = \{x_i, x_j\}|_{a_0=\dots=b_2=0}, \quad \{x_i, x_j\}_{k,1} = \frac{\partial \{x_i, x_j\}}{\partial a_k}, \quad \{x_i, x_j\}_{k,2} = \frac{\partial \{x_i, x_j\}}{\partial b_k}.$$

## 2.3 Construction in the case of odd $n$

**Proposition 1'.** The following formula

$$\{\phi, \psi\} = (n\lambda_{od} + g_2 - g_1 + \frac{1}{2}b_2(f_1 - f_2))(\phi_1\psi_2 - \psi_1\phi_2) + \phi_1D(\psi_2) + \phi_2D(\psi_1) - \psi_1D(\phi_2) - \psi_2D(\phi_1) \quad (2.12)$$

defines a quadratic Poisson bracket on the polynomial ring  $S^*{\mathcal F}_n = \mathbb{C}[x_0, x_2, \dots, x_n]$  where  $n$  is odd. Here  $\phi, \psi \in {\mathcal F}_n$  and  $\{\phi, \psi\} \in S^2{\mathcal F}_n$ . This Poisson bracket is linear with respect to  $c$  and coefficients  $a_0, \dots, a_3, b_0, \dots, b_2$  of polynomials  $P_{od}$ ,  $Q$  and, therefore, can be written in the form  $\{\cdot, \cdot\} = \{\cdot, \cdot\}_0 + \sum_{i=0}^3 a_i \{\cdot, \cdot\}_{i,1} + \sum_{j=0}^2 b_j \{\cdot, \cdot\}_{j,2} + c \{\cdot, \cdot\}_3$  where  $\{\cdot, \cdot\}_0, \{\cdot, \cdot\}_{i,1}, \{\cdot, \cdot\}_{j,2}, \{\cdot, \cdot\}_3$  are pairwise compatible. Therefore, for each odd  $n$  we have constructed nine compatible quadratic Poisson brackets in  $n$  variables.

**Proof.** The Jacobi identity is a consequence of a functional construction described in the next section. Let us check linearity with respect to  $c$  and coefficients of  $P_{od}$ ,  $Q$ . Each of  $\phi, \psi \in \mathcal{F}_n \subset \mathcal{F}_{ev}$  can be of the form  $R(f)$  or  $R(f)g$  where  $R$  is a polynomial. Therefore, we have three cases:

**Case 1.** Let  $\phi = R(f)$ ,  $\psi = T(f)$ . We have

$$\begin{aligned}
\{\phi, \psi\} &= \{R(f), T(f)\} \\
&= (n\lambda_{od} + g_2 - g_1 + \frac{1}{2}b_2(f_1 - f_2))(R(f_1)T(f_2) - R(f_2)T(f_1)) + \\
&\quad R(f_1)D(T(f_2)) + R(f_2)D(T(f_1)) \\
&\quad - T(f_1)D(R(f_2)) - T(f_2)D(R(f_1)) \\
&= n \frac{R(f_1)T(f_2) - R(f_2)T(f_1)}{f_1 - f_2} \left( (f_1 + c)g_1 + (f_2 + c)g_2 - \frac{1}{2}Q(f_1) - \frac{1}{2}Q(f_2) \right) + \\
&\quad (g_2 - g_1 + \frac{1}{2}b_2(f_1 - f_2))(R(f_1)T(f_2) - R(f_2)T(f_1)) + \\
&\quad (R(f_2)T'(f_1) - T(f_2)R'(f_1))(2(f_1 + c)g_1 - Q(f_1)) + \\
&\quad (R(f_1)T'(f_2) - T(f_1)R'(f_2))(2(f_2 + c)g_2 - Q(f_2))
\end{aligned}$$

**Case 2.** Let  $\phi = R(f)$ ,  $\psi = T(f)g$ . We have

$$\begin{aligned}
\{\phi, \psi\} &= \{R(f), T(f)\} \\
&= (n\lambda_{od} + g_2 - g_1 + \frac{1}{2}b_2(f_1 - f_2))(R(f_1)T(f_2)g_2 - R(f_2)T(f_1)g_1) + \\
&\quad R(f_1)D(T(f_2)g_2) + R(f_2)D(T(f_1)g_1) \\
&\quad - T(f_1)g_1D(R(f_2)) - T(f_2)g_2D(R(f_1)) \\
&= \frac{n}{2} \frac{R(f_1)T(f_2) - R(f_2)T(f_1)}{f_1 - f_2} (f_1 + f_2 + 2c)g_1g_2 - \\
&\quad \frac{n}{2} \frac{Q(f_1) - Q(f_2)}{f_1 - f_2} (R(f_1)T(f_2)g_2 + R(f_2)T(f_1)g_1) + \\
&\quad \frac{n-2}{2} (R(f_1)T(f_2) + R(f_2)T(f_1))g_1g_2 + \\
&\quad n \frac{R(f_1)T(f_2)P_{od}(f_2) - R(f_2)T(f_1)P_{od}(f_1)}{f_1 - f_2} + \\
&\quad \frac{1}{2}b_2(f_1 - f_2)(R(f_1)T(f_2)g_2 - R(f_2)T(f_1)g_1) \\
&\quad + R(f_1)T(f_2)(P'_{od}(f_2) + Q'(f_2)g_2) + R(f_2)T(f_1)(P'_{od}(f_1) + Q'(f_1)g_1) \\
&\quad + R'(f_1)T(f_2)(Q(f_1) - 2(f_1 + c)g_1)g_2 + R'(f_2)T(f_1)(Q(f_2) - 2(f_2 + c)g_2)g_1 \\
&\quad + R(f_1)T'(f_2)(2P_{od}(f_2) + Q(f_2)g_2) + R(f_2)T'(f_1)(2P_{od}(f_1) + Q(f_1)g_1).
\end{aligned}$$

Here we used  $\lambda_{od}(R(f_1)T(f_2)g_2 - R(f_2)T(f_1)g_1) =$

$$(g_1 + g_2) \frac{R(f_1)T(f_2) - R(f_2)T(f_1)}{2(f_1 - f_2)} \lambda_{od}(f_1 - f_2) - \frac{1}{2}(R(f_1)T(f_2) + R(f_2)T(f_1))\lambda_{od}(g_1 - g_2).$$

**Case 3.** Let  $\phi = R(f)g$ ,  $\psi = T(f)g$ . We have

$$\begin{aligned}
\{\phi, \psi\} &= \{R(f), T(f)\} \\
&= (n\lambda_{od} + g_2 - g_1 + \frac{1}{2}b_2(f_1 - f_2))((R(f_1)T(f_2) - R(f_2)T(f_1))g_1g_2) + \\
&\quad R(f_1)g_1D(T(f_2)g_2) + R(f_2)g_2D(T(f_1)g_1) \\
&\quad - T(f_1)g_1D(R(f_2)g_2) - T(f_2)g_2D(R(f_1)g_1) \\
&= n\frac{R(f_1)T(f_2) - R(f_2)T(f_1)}{f_1 - f_2}((f_1 + c)g_1^2g_2 + (f_2 + c)g_1g_2^2 - \frac{1}{2}Q(f_1)g_1g_2 - \frac{1}{2}Q(f_2)g_1g_2) + \\
&\quad (R(f_1)T(f_2) - R(f_2)T(f_1))(\frac{1}{2}b_2(f_1 - f_2)g_1g_1 + g_1g_2^2 - g_1^2g_2) \\
&\quad + (R(f_2)T'(f_1) - T(f_2)R'(f_1))(2(f_1 + c)g_1 - Q(f_1))g_1g_2 \\
&\quad + (R(f_1)T'(f_2) - T(f_1)R'(f_2))(2(f_2 + c)g_2 - Q(f_2))g_1g_2 \\
&\quad + (R(f_1)T(f_2) - R(f_2)T(f_1))(g_1D(g_2) - D(g_1)g_2) \\
&= n\frac{R(f_1)T(f_2) - R(f_2)T(f_1)}{f_1 - f_2}(P_{od}(f_1)g_2 + g_1P_{od}(f_2) + \frac{1}{2}Q(f_1)g_1g_2 + \frac{1}{2}Q(f_2)g_1g_2) \\
&\quad + \frac{b_2}{2}(R(f_1)T(f_2) - R(f_2)T(f_1))(f_1 - f_2)g_1g_2 \\
&\quad + (R(f_2)T'(f_1) - T(f_2)R'(f_1))(2P_{od}(f_1) + Q(f_1)g_1)g_2 \\
&\quad + (R(f_1)T'(f_2) - T(f_1)R'(f_2))(2P_{od}(f_2) + Q(f_2)g_2)g_1 \\
&\quad + (R(f_1)T(f_2) - R(f_2)T(f_1))(P'_{od}(f_2)g_1 + Q'(f_2)g_1g_2 - P'_{od}(f_1)g_2 - Q'(f_1)g_1g_2)
\end{aligned}$$

In these computations we use formulas (2.4) and (2.6) where  $f, g$  are replaced by  $f_1, g_1$  or  $f_2, g_2$ . Note that in each case we obtain an expression for  $\{\phi, \psi\}$  that is linear non-homogeneous in  $P_{od}$ ,  $Q$  and bi-linear non-homogeneous in  $g_1, g_2$ . Using identifications (2.10) we can write each of these expressions as a quadratic homogeneous polynomial in  $x_0, x_2, x_3, \dots, x_n$  with coefficients linear in  $a_0, \dots, a_3, b_0, \dots, b_2, c$ .

**Remark 1'.** Let us describe an algorithm for the computation of  $\{x_i, x_j\}$  as a quadratic polynomial in  $x_0, x_2, x_3, \dots, x_n$ . One uses the formulas for  $\{\phi, \psi\}$  from the proof of the proposition 1' where  $R(f) = f^i$ ,  $T(f) = f^j$  and  $P_{od}$ ,  $Q$  are given by (2.2). For the computation of  $\{x_{2i}, x_{2j}\}$  the formula of case 1 is used, for the computation of  $\{x_{2i}, x_{2j+3}\}$  the formula of case 2 is used, and for the computation of  $\{x_{2i+3}, x_{2j+3}\}$  the formula of case 3 is used. These formulas give a polynomial in  $f_1, f_2, g_1, g_2$  linear in  $g_1, g_2$ . This polynomial is symmetric under transformations  $f_1 \leftrightarrow f_2$ ,  $g_1 \leftrightarrow g_2$ . By using identifications (2.10) the expressions can be written as a polynomial quadratic in  $x_0, x_2, x_3, \dots, x_n$ .

**Remark 2'.** In all cases of the above remark  $\{x_i, x_j\}$  is linear non-homogeneous with respect to  $c$  and the seven coefficients  $a_0, \dots, b_2$  of polynomials  $P_{od}$ ,  $Q$ . Therefore, nine compatible Poisson brackets can be obtained in the following way:

$$\{x_i, x_j\}_0 = \{x_i, x_j\}|_{a_0 = \dots = b_2 = c = 0}, \quad \{x_i, x_j\}_{k,1} = \frac{\partial \{x_i, x_j\}}{\partial a_k}, \quad \{x_i, x_j\}_{k,2} = \frac{\partial \{x_i, x_j\}}{\partial b_k}, \quad \{x_i, x_j\}_3 = \frac{\partial \{x_i, x_j\}}{\partial c}.$$

**Remark 3.** If we replace  $n$  in the formulas (2.11), (2.12) by an arbitrary constant  $\alpha$ , then these formulas still define Poisson brackets on the polynomial algebra  $\mathbb{C}[x_0, x_2, x_3, \dots]$ . However,  $\mathbb{C}[x_0, x_2, x_3, \dots, x_n] \subset \mathbb{C}[x_0, x_2, x_3, \dots]$  is closed with respect to these brackets only if  $\alpha = n$ .

### 3 Functional construction

#### 3.1 General constructions

Recall a general construction of associative algebras and Poisson structures [5]. Let  $\lambda(x, y)$  be a meromorphic function in two variables. We construct an associative algebra  $A_\lambda$  through:

$$A_\lambda = \mathbb{C} \oplus F_1 \oplus F_2 \oplus F_3 \oplus \dots$$

where  $F_m$  is the space of symmetric meromorphic functions in  $m$  variables and a product  $f \star g \in F_{a+b}$  of  $f \in F_a$ ,  $g \in F_b$  is defined by:

$$f \star g(z_1, \dots, z_{a+b}) = \frac{1}{a!b!} \sum_{\sigma \in S_{a+b}} f(z_{\sigma_1}, \dots, z_{\sigma_a}) g(z_{\sigma_{a+1}}, \dots, z_{\sigma_{a+b}}) \prod_{1 \leq p \leq a, a+1 \leq q \leq a+b} \lambda(z_{\sigma_p}, z_{\sigma_q}).$$

Note that this formula defines an associative product for an arbitrary function  $\lambda$  and this product is non-commutative if  $\lambda$  is not symmetric. In particular, if  $\lambda(x, y) = 1 + \frac{1}{2}\epsilon\mu(x, y) + o(\epsilon)$ , then we obtain a Poisson algebra. Assume  $\mu(x, y) = -\mu(y, x)$ . For  $f, g \in F_1$  we get the following formulas for the associative commutative product  $fg \in F_2$  and the Poisson bracket:

$$fg(x, y) = f(x)g(y) + g(x)f(y), \quad \{f, g\} = (f(x)g(y) - g(x)f(y))\mu(x, y).$$

If  $\lambda = \lambda(x - y)$ , then the formula for the product can be deformed in the following way:

$$f \star g(z_1, \dots, z_{a+b}) = \frac{1}{a!b!} \sum_{\sigma \in S_{a+b}} f(z_{\sigma_1}, \dots, z_{\sigma_a}) g(z_{\sigma_{a+1}} + ap, \dots, z_{\sigma_{a+b}} + ap) \prod_{1 \leq p \leq a, a+1 \leq q \leq a+b} \lambda(z_{\sigma_p} - z_{\sigma_q})$$

where  $p$  is an arbitrary constant. The corresponding formula for the Poisson brackets (if we set  $p = \epsilon\alpha$ ) is:

$$\{f, g\} = (f(x)g(y) - g(x)f(y))\mu(x - y) + \alpha(f(x)g'(y) + f(y)g'(x) - g(x)f'(y) - g(y)f'(x)).$$

We will need the following generalization of the last formula:

$$\begin{aligned} \{f, g\} = & (f(x)g(y) - g(x)f(y))(\mu(x - y) + \nu(x) - \nu(y)) \\ & + \alpha(f(x)g'(y) + f(y)g'(x) - g(x)f'(y) - g(y)f'(x)). \end{aligned} \quad (3.13)$$

Here  $\nu$  is an arbitrary function. This formula is obtained from the previous one by transformation  $f \rightarrow \kappa f$ ,  $g \rightarrow \kappa g$ ,  $\{f, g\} \rightarrow \kappa^2 \{f, g\}$  where  $\alpha\kappa' = -\nu$ .

Note that the algebra  $A_\lambda$  is very large. One can construct associative algebras (and the corresponding Poisson algebras) of a reasonable size by a suitable choice of spaces  $F_\alpha$  and function  $\lambda$ . For example, the algebra  $Q_n(\tau, \eta)$  and the Poisson algebra  $q_n(\tau)$  can be constructed in this way [5]. See [7] for the functional construction of a wider class of Poisson algebras.

### 3.2 The case of even $n$

It is clear that equation (2.3) defines an elliptic curve in  $\mathbb{C}^2$  with coordinates  $f, g$ . Therefore, one can find elliptic functions  $f = f(z), g = g(z)$  such that

$$g(z)^2 = P_{ev}(f(z)) + Q(f(z))g(z). \quad (3.14)$$

Moreover, one can assume (see (2.5))

$$f'(z) = 2g(z) - Q(f(z)), \quad g'(z) = P'_{ev}(f(z)) + Q'(f(z))g(z). \quad (3.15)$$

Note that elliptic functions  $f(z), g(z)$  have a form:

$$f(z) = c_1 + c_2\zeta(z, \tau) + c_3\zeta(z - u, \tau), \quad g(z) = c_4 + c_5\zeta(z, \tau) + c_6\zeta(z - u, \tau) + c_7\zeta'(z, \tau) + c_8\zeta'(z - u, \tau).$$

Here  $\zeta(z, \tau)$  is the Weierstrass elliptic function,  $\tau$  is a modular parameter and constants  $c_1, \dots, c_8, u, \tau$  are determined by  $a_0, \dots, b_2$ . There exists an elliptic function in two variables  $\mu_{ev}(z_1, z_2)$  such that

$$(f(z_1) - f(z_2))\mu_{ev}(z_1, z_2) = g(z_1) + g(z_2) - \frac{1}{2}Q(f(z_1)) - \frac{1}{2}Q(f(z_2)), \quad (3.16)$$

$$(g(z_1) - g(z_2))\mu_{ev}(z_1, z_2) = \frac{P_{ev}(f(z_1)) - P_{ev}(f(z_2))}{f(z_1) - f(z_2)} + \frac{Q(f(z_1)) - Q(f(z_2))}{2(f(z_1) - f(z_2))}(g(z_1) + g(z_2)).$$

This function has the form

$$\mu_{ev}(z_1, z_2) = \zeta(z_1 - z_2, \tau) + k_1\zeta(z_1, \tau) + k_2\zeta(z_1 - u, \tau) - k_1\zeta(z_2, \tau) - k_2\zeta(z_2 - u, \tau)$$

for some constants  $k_1, k_2$  such that  $k_1 + k_2 = 1$ .

Let  $\mathcal{F}_n$  be the space of elliptic functions in one variable with periods 1 and  $\tau$ , holomorphic outside  $z = 0, u$  modulo periods and having poles of order not larger than  $n$  at  $z = 0, u$ . It is clear that  $\{e_0(z), e_2(z), e_3(z), e_4(z), \dots, e_n(z)\}$  is a basis of the linear space  $\mathcal{F}_n$  where we define

$$e_{2i}(z) = f(z)^i, \quad e_{2i+3}(z) = f(z)^i g(z), \quad i = 0, 1, 2, \dots$$

We will identify  $S^m \mathcal{F}_n$  with the space of symmetric elliptic functions in  $m$  variables  $\{h(z_1, \dots, z_m)\}$  holomorphic if  $z_k \neq 0, u$  modulo periods and having poles of order not larger than  $n$  at  $z_k = 0, u$ . We construct a bilinear operator  $\{\cdot, \cdot\} : \Lambda^2 \mathcal{F}_n \rightarrow S^2 \mathcal{F}_n$  as follows: for  $\phi, \psi \in \mathcal{F}_n$  we set

$$\begin{aligned} \{\phi, \psi\}(z_1, z_2) &= n\mu_{ev}(z_1, z_2)(\phi(z_1)\psi(z_2) - \psi(z_1)\phi(z_2)) \\ &\quad + \phi(z_1)\psi'(z_2) + \phi(z_2)\psi'(z_1) - \psi(z_1)\phi'(z_2) - \psi(z_2)\phi'(z_1). \end{aligned} \quad (3.17)$$

**Proposition 2.** The formula (3.17) defines a Poisson structure on the polynomial algebra  $S^* \mathcal{F}_n$ . This Poisson bracket is linear with respect to coefficients  $a_0, \dots, a_4, b_0, \dots, b_2$  of polynomials  $P_{ev}, Q$  and, therefore, can be written in the form  $\{\cdot, \cdot\} = \{\cdot, \cdot\}_0 + \sum_{i=0}^4 a_i \{\cdot, \cdot\}_{i,1} + \sum_{j=0}^2 b_j \{\cdot, \cdot\}_{j,2}$  where  $\{\cdot, \cdot\}_0, \{\cdot, \cdot\}_{i,1}, \{\cdot, \cdot\}_{j,2}$  are pairwise compatible. Therefore, for each even  $n$  we have constructed nine compatible quadratic Poisson brackets in  $n$  variables.

**Proof.** This is just a reformulation of the Proposition 1. Formula (3.17) is a special case of (3.13) and therefore the Jacobi identity for (3.17) is satisfied. One can check straightforwardly that if  $\phi, \psi \in \mathcal{F}_n$ , then  $\{\phi, \psi\}(z_1, z_2)$  given by (3.17) is a symmetric elliptic function in two variables  $z_1, z_2$  having poles of order not larger than  $n$  at  $z_1, z_2 = 0, u$  and therefore  $\{\phi, \psi\}(z_1, z_2) \in S^2 \mathcal{F}_n$ .

### 3.3 The case of odd $n$

It is clear that equation (2.4) defines an elliptic curve in  $\mathbb{C}^2$  with coordinates  $f, g$ . Therefore, one can find elliptic functions  $f = f(z), g = g(z)$  such that

$$(f(z) + c)g(z)^2 = P_{od}(f(z)) + Q(f(z))g(z). \quad (3.18)$$

Moreover, one can assume (see (2.6))

$$f'(z) = 2(f(z) + c)g(z) - Q(f(z)), \quad g'(z) = P'_{od}(f(z)) + Q'(f(z))g(z) - g(z)^2. \quad (3.19)$$

Note that elliptic functions  $f(z), g(z)$  have a form:

$$f(z) = c_1 + c_2\zeta(z, \tau) + c_3\zeta(z - u, \tau), \quad g(z) = c_4 + c_5\zeta(z, \tau) + c_6\zeta(z - u, \tau) + c_7\zeta(z - v, \tau).$$

Here  $\zeta(z, \tau)$  is the Weierstrass elliptic function,  $\tau$  is a modular parameter and constants  $c_1, \dots, c_7, u, v, \tau$  are determined by  $a_0, \dots, b_2, c$ . There exists an elliptic function in two variables  $\mu_{od}(z_1, z_2)$  such that

$$(f(z_1) - f(z_2))\mu_{od}(z_1, z_2) = (f(z_1) + c)g(z_1) + (f(z_2) + c)g(z_2) - \frac{1}{2}Q(f(z_1)) - \frac{1}{2}Q(f(z_2)), \quad (3.20)$$

$$(g(z_1) - g(z_2))\mu_{od}(z_1, z_2) = \frac{P(f(z_1)) - P(f(z_2))}{f(z_1) - f(z_2)} + \frac{Q(f(z_1)) - Q(f(z_2))}{2(f(z_1) - f(z_2))}(g(z_1) + g(z_2)) - g(z_1)g(z_2).$$

This function has the form

$$\mu_{od}(z_1, z_2) = \zeta(z_1 - z_2, \tau) + k_1\zeta(z_1, \tau) + k_2\zeta(z_1 - u, \tau) - k_1\zeta(z_2, \tau) - k_2\zeta(z_2 - u, \tau)$$

for some constants  $k_1, k_2$  such that  $k_1 + k_2 = 1$ .

Let  $\mathcal{F}_n$  be the space of elliptic functions in one variable with periods 1 and  $\tau$ , holomorphic outside  $z = 0, u, v$  modulo periods and having poles of order not larger than  $n$  at  $z = 0, u$  and not larger than one at  $z = v$ . It is clear that  $\{e_0(z), e_2(z), e_3(z), e_4(z), \dots, e_n(z)\}$  is a basis of the linear space  $\mathcal{F}_n$  where we define

$$e_{2i}(z) = f(z)^i, \quad e_{2i+3}(z) = f(z)^i g(z), \quad i = 0, 1, 2, \dots$$

We will identify  $S^m \mathcal{F}_n$  with the space of symmetric elliptic functions in  $m$  variables  $\{h(z_1, \dots, z_m)\}$  holomorphic if  $z_k \neq 0, u, v$  modulo periods and having poles of order not larger than  $n$  at  $z_k = 0, u$  and not larger than one at  $z_k = v$ . We construct a bilinear operator  $\{\cdot, \cdot\} : \Lambda^2 \mathcal{F}_n \rightarrow S^2 \mathcal{F}_n$  as follows: for  $\phi, \psi \in \mathcal{F}_n$  we set

$$\begin{aligned} \{\phi, \psi\}(z_1, z_2) &= (n\mu_{od}(z_1, z_2) + g(z_2) - g(z_1) + \frac{1}{2}b_2(f(z_1) - f(z_2)))(\phi(z_1)\psi(z_2) - \psi(z_1)\phi(z_2)) \\ &\quad + \phi(z_1)\psi'(z_2) + \phi(z_2)\psi'(z_1) - \psi(z_1)\phi'(z_2) - \psi(z_2)\phi'(z_1). \end{aligned} \quad (3.21)$$

**Proposition 2'.** The formula (3.21) defines a Poisson structure on the polynomial algebra  $S^* \mathcal{F}_n$ . This Poisson bracket is linear with respect to  $c$  and coefficients  $a_0, \dots, a_3, b_0, \dots, b_2$  of polynomials  $P_{od}, Q$  and, therefore, can be written in the form  $\{\cdot, \cdot\} = \{\cdot, \cdot\}_0 + \sum_{i=0}^3 a_i \{\cdot, \cdot\}_{i,1} + \sum_{j=0}^2 b_j \{\cdot, \cdot\}_{j,2} + c \{\cdot, \cdot\}_3$  where

$\{\cdot, \cdot\}_0, \{\cdot, \cdot\}_{i,1}, \{\cdot, \cdot\}_{j,2}, \{\cdot, \cdot\}_3$  are pairwise compatible. Therefore, for each odd  $n$  we have constructed nine compatible quadratic Poisson brackets in  $n$  variables.

**Proof.** This is just a reformulation of the Proposition 1'. Formula (3.21) is a special case of (3.13) and therefore the Jacobi identity for (3.21) is satisfied. One can check straightforwardly that if  $\phi, \psi \in \mathcal{F}_n$ , then  $\{\phi, \psi\}(z_1, z_2)$  given by (3.21) is a symmetric elliptic function in two variables  $z_1, z_2$  having poles of order not larger than  $n$  at  $z_1, z_2 = 0, u$  and not larger than one at  $z_1, z_2 = v$  and therefore  $\{\phi, \psi\}(z_1, z_2) \in S^2 \mathcal{F}_n$ .

## 4 Symplectic leaves and Casimir elements

For  $p, n \in \mathbb{N}$  we denote by  $b_{p,n}$  the Poisson algebra spanned by the elements

$$\{h(f_1, g_1, \dots, f_p, g_p) e_1^{\alpha_1} \dots e_p^{\alpha_p}; \alpha_1, \dots, \alpha_p \in \mathbb{Z}_{\geq 0}\}$$

as a linear space, where  $h$  is a rational function,  $f_i, g_i$  for each  $i = 1, \dots, p$  are subject to relation (2.3) if  $n$  is even and (2.4) if  $n$  is odd where  $f, g$  are replaced by  $f_i, g_i$ . In other words<sup>3</sup>,  $h \in \text{Frac}(\otimes^p \mathcal{F})$ . A Poisson bracket on  $b_{p,n}$  is defined as follows:

$$\{f_i, e_j\} = -D(f_i)e_j, \quad \{g_i, e_j\} = -D(g_i)e_j, \quad \{f_i, e_i\} = \frac{n-2}{2}D(f_i)e_i, \quad \{g_i, e_i\} = \frac{n-2}{2}D(g_i)e_i$$

where  $i \neq j$ ,  $D$  is defined by (2.5) for even  $n$  and (2.6) for odd  $n$ . We also assume

$$\{e_i, e_j\} = n\lambda_{i,j,ev}e_i e_j$$

if  $n$  is even and

$$\{e_i, e_j\} = \left( n\lambda_{i,j,od} + \frac{1}{2}b_2(f_i - f_j) + g_j - g_i \right) e_i e_j$$

if  $n$  is odd. Here  $\lambda_{i,j,ev}$  (resp.  $\lambda_{i,j,od}$ ) is given by (2.7) (resp. (2.8)) where  $f_1, g_1, f_2, g_2$  are replaced by  $f_i, g_i, f_j, g_j$  correspondingly. All brackets between  $f_i, g_j$  are zero.

Let us define a linear map  $\phi_p : \mathcal{F}_n \rightarrow b_{p,n}$  by the formula

$$\phi_p(x_{2j}) = \sum_{i=1}^p f_i^j e_i, \quad \phi_p(x_{2j+3}) = \sum_{i=1}^p f_i^j g_i e_i.$$

There is a unique extension of this map to the homomorphism of commutative algebras  $S^*(\mathcal{F}_n) \rightarrow b_{p,n}$  which we also denote by  $\phi_p$ .

**Proposition 3.** The map  $\phi_p : S^*(\mathcal{F}_n) \rightarrow b_{p,n}$  is a homomorphism of Poisson algebras.

**Proof.** One can check straightforwardly that  $\phi_p(\{r, s\}) = \sum_{i,j=1}^p \{r, s\}_{i,j} e_i e_j$  where  $r, s$  are arbitrary elements from  $\mathcal{F}_n$  and  $\{r, s\}_{i,j}$  is obtained from  $\{r, s\} \in \mathcal{F}_n \otimes \mathcal{F}_n$  replacing  $f_1, g_1, f_2, g_2$  by  $f_i, g_i, f_j, g_j$  correspondingly. This implies the proposition.

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<sup>3</sup>Recall that  $\mathcal{F} = \mathcal{F}_{ev}$  if  $n$  is even and  $\mathcal{F} = \mathcal{F}_{od}$  if  $n$  is odd.

It is known [5, 6] that if  $2p < n$ , then the map  $\phi_p$  defines a  $2p$  dimensional symplectic leaf of the Poisson algebra  $S^*(\mathcal{F}_n)$ . Moreover, central elements of the Poisson algebra  $S^*(\mathcal{F}_n)$  belong to  $\ker \phi_p$  for  $2p < n$ . One can check that for  $p = \frac{n}{2} - 1$  for even  $n$  (resp.  $p = \frac{n-1}{2}$  for odd  $n$ ) the ideal  $\ker \phi_p$  is generated by two elements of degree  $\frac{n}{2}$  (resp. by one element of degree  $n$ ). We denote these elements by  $C_0^{\frac{n}{2}}, C_1^{\frac{n}{2}}$  if  $n$  is even (resp. by  $C^n$  if  $n$  is odd). The center of the Poisson algebra  $S^*(\mathcal{F}_n)$  is a polynomial algebra generated by  $C_0^{\frac{n}{2}}, C_1^{\frac{n}{2}}$  if  $n$  is even (resp. by  $C^n$  if  $n$  is odd). Let us describe these elements explicitly (see also [6]).

Let  $n$  be even. We define elements  $x_i, i \in \mathbb{Z}$  by the formula (2.9). It is clear that  $x_i \in \text{Frac}(\mathcal{F}_{ev})$  if  $i = 1, -1, -2, \dots$  and  $x_0, x_2, \dots, x_n \in \mathcal{F}_n$ . If  $\frac{n}{2}$  is even we set<sup>4</sup>

$$\begin{aligned} C_0^{\frac{n}{2}} &= \det \left( (x_0, x_2, \dots, x_{\frac{n}{2}})^t (x_0, x_2, \dots, x_{\frac{n}{2}}) \right), \\ C_1^{\frac{n}{2}} &= \det \left( (x_0, x_1, x_2, \dots, x_{\frac{n}{2}-1})^t (x_2, \dots, x_{\frac{n}{2}+1}) \right) \\ &\quad - \det \left( (x_0, x_1, x_2, \dots, x_{\frac{n}{2}-2}, x_{\frac{n}{2}})^t (x_2, \dots, x_{\frac{n}{2}}, a_4 x_{\frac{n}{2}+2} + b_2 x_{\frac{n}{2}+1}) \right) \\ &\quad - \det \left( (a_0 x_{-2} + b_0 x_1, x_0, x_2, \dots, x_{\frac{n}{2}-1})^t (x_0, x_2, x_4, \dots, x_{\frac{n}{2}+1}) \right) \\ &\quad + \det \left( (a_0 x_{-2} + b_0 x_1, x_0, x_2, \dots, x_{\frac{n}{2}-2}, x_{\frac{n}{2}})^t (x_0, x_2, x_4, \dots, x_{\frac{n}{2}}, a_4 x_{\frac{n}{2}+2} + b_2 x_{\frac{n}{2}+1}) \right) \end{aligned}$$

and if  $\frac{n}{2}$  is odd

$$\begin{aligned} C_0^{\frac{n}{2}} &= \det \left( (x_0, x_2, \dots, x_{\frac{n}{2}})^t (x_0, x_2, \dots, x_{\frac{n}{2}}) \right) - \det \left( (x_0, x_2, \dots, x_{\frac{n}{2}-1}, x_{\frac{n}{2}+1})^t (x_0, x_2, \dots, x_{\frac{n}{2}-1}, b_2 x_{\frac{n}{2}} + a_4 x_{\frac{n}{2}+1}) \right), \\ C_1^{\frac{n}{2}} &= \det \left( (x_2, \dots, x_{\frac{n}{2}+1})^t (x_0, x_1, x_2, \dots, x_{\frac{n}{2}-1}) \right) - \det \left( (x_{-2}, x_0, x_2, \dots, x_{\frac{n}{2}-1})^t (a_0 x_0 + b_0 x_3, x_2, x_4, \dots, x_{\frac{n}{2}+1}) \right). \end{aligned}$$

In these formulas we use the product in the algebra  $\text{Frac}(\mathcal{F}_{ev})$  for computing products of vector components. Therefore entries of our matrices of the form  $v^t w$  are linear combinations of  $x_i \in \text{Frac}(\mathcal{F}_{ev})$ ,  $i \in \mathbb{Z}$ . On the other hand, for computing determinants we use the product in  $S^*(\text{Frac}(\mathcal{F}_{ev}))$ . So these determinants are polynomials of degree  $\frac{n}{2}$  in  $x_i, i \in \mathbb{Z}$ .<sup>5</sup> Moreover, it turns out that in our linear combinations of determinants all terms with  $x_i \notin \{x_0, x_2, \dots, x_n\}$  cancel out and  $C_0^{\frac{n}{2}}, C_1^{\frac{n}{2}}$  are polynomials in  $x_0, x_2, \dots, x_n$  of degree  $\frac{n}{2}$ .

Let  $n$  be odd. We define elements  $y_i \in \text{Frac}(\mathcal{F}_{od})$  for  $i \in \mathbb{Z}$  by

$$y_{2i} = (f + c)^i, \quad y_{2i+3} = (f + c)^i g.$$

It is clear that  $y_i \in \text{Frac}(\mathcal{F}_{od})$  if  $i = 1, -1, -2, \dots$  and  $y_0, y_2, \dots, y_n \in \mathcal{F}_n$ . Moreover,  $y_0, y_2, \dots, y_n$  can be

<sup>4</sup>These formulas work for  $n > 4$ . If  $n = 4$  we set

$$\begin{aligned} C_0^2 &= \det \left( (x_0, x_2)^t (x_0, x_2) \right), \\ C_1^2 &= \det \left( (x_0, x_1)^t (x_2, x_3) \right) - \det \left( (x_0, x_2)^t (x_2, a_4 x_4 + b_2 x_3) \right) - \det \left( (a_0 x_{-2} + b_0 x_1, x_0)^t (x_0, x_2) \right) \end{aligned}$$

<sup>5</sup>For example,  $\det((x_0, x_2)^t (x_0, x_2)) = \det((1, f)^t (1, f)) = \det \begin{pmatrix} 1 & f \\ f & f^2 \end{pmatrix} = \det \begin{pmatrix} x_0 & x_2 \\ x_2 & x_4 \end{pmatrix} = x_0 x_4 - x_2^2$ .

written as linear combinations of  $x_0, x_2, \dots, x_n \in \mathcal{F}_n$ . If  $\frac{n+1}{2}$  is even we set<sup>6</sup>

$$\begin{aligned}\tilde{C}_0^{\frac{n+1}{2}} &= \det \left( (y_0, y_2, y_4, \dots, y_{\frac{n+3}{2}})^t (y_{-2}, y_0, y_2, \dots, y_{\frac{n-1}{2}}) \right), \\ \tilde{C}_1^{\frac{n+1}{2}} &= \det \left( (y_0, y_2, \dots, y_{\frac{n-1}{2}}, y_{\frac{n+3}{2}})^t (y_0, y_2, \dots, y_{\frac{n-1}{2}}, y_{\frac{n+3}{2}}) \right) \\ &\quad - \det \left( (y_0, y_2, \dots, y_{\frac{n+1}{2}})^t (y_0, y_2, \dots, y_{\frac{n-1}{2}}, a_3 y_{\frac{n+1}{2}} + b_2 y_{\frac{n+3}{2}}) \right) \\ &\quad - \det \left( (Q(-c)y_0, y_2, \dots, y_{\frac{n-1}{2}}, y_{\frac{n+3}{2}})^t (y_{-2}, y_0, y_2, y_4, \dots, y_{\frac{n-1}{2}}, y_{\frac{n+3}{2}}) \right) \\ &\quad + \det \left( (Q(-c)y_0, y_2, \dots, y_{\frac{n+1}{2}})^t (y_{-2}, y_0, y_2, y_4, \dots, y_{\frac{n-1}{2}}, a_3 y_{\frac{n+1}{2}} + b_2 y_{\frac{n+3}{2}}) \right)\end{aligned}$$

and if  $\frac{n+1}{2}$  is odd

$$\begin{aligned}\tilde{C}_0^{\frac{n+1}{2}} &= \det \left( (y_0, y_2, y_4, \dots, y_{\frac{n+1}{2}}, y_{\frac{n+5}{2}})^t (y_{-2}, y_0, y_2, \dots, y_{\frac{n-3}{2}}, y_{\frac{n+1}{2}}) \right) \\ &\quad - \det \left( (y_0, y_2, y_4, \dots, y_{\frac{n+3}{2}})^t (y_{-2}, y_0, y_2, \dots, y_{\frac{n-3}{2}}, b_2 y_{\frac{n+1}{2}} + a_3 y_{\frac{n-1}{2}}) \right), \\ \tilde{C}_1^{\frac{n+1}{2}} &= \det \left( (y_0, y_2, \dots, y_{\frac{n+1}{2}})^t (y_0, y_2, \dots, y_{\frac{n+1}{2}}) \right) - \det \left( (y_{-2}, y_0, y_2, y_4, \dots, y_{\frac{n+1}{2}})^t (Q(-c)y_0, y_2, \dots, y_{\frac{n+1}{2}}) \right).\end{aligned}$$

In these formulas we use the product in the algebra  $\text{Frac}(\mathcal{F}_{od})$  for computing products of vector components. Therefore entries of our matrices of the form  $v^t w$  are linear combinations of  $y_i \in \text{Frac}(\mathcal{F}_{od})$ ,  $i \in \mathbb{Z}$ . On the other hand, for computing determinants we use the product in  $S^*(\text{Frac}(\mathcal{F}_{od}))$ . So these determinants are polynomials of degree  $\frac{n+1}{2}$  in  $y_i$ ,  $i \in \mathbb{Z}$ . Moreover, it turns out that in our linear combinations of determinants all terms with  $y_i \notin \{y_{-2}, y_0, y_2, \dots, y_n\}$  cancel out and  $\tilde{C}_0^{\frac{n+1}{2}}, \tilde{C}_1^{\frac{n+1}{2}}$  are polynomials in  $y_{-2}, y_0, y_2, \dots, y_n$  of degree  $\frac{n+1}{2}$ . These polynomials are linear in  $y_{-2}$  and therefore can be written as  $\tilde{C}_i^{\frac{n+1}{2}} = A_i + B_i y_{-2}$ ,  $i = 0, 1$  where  $A_i, B_i$  are polynomials in  $y_0, y_2, \dots, y_n$ . We set  $C^n = A_0 B_1 - A_1 B_0$ .

## 5 Conclusion

In this paper we have constructed nine pairwise compatible quadratic Poisson structures on a linear space of arbitrary dimension. It seems that this family of Poisson structures is maximal if the dimension of linear space is larger than four. We think that the following problems deserve further investigation:

- Study the differential and algebraic geometry of these compatible Poisson structures to explain geometrically why these structures exist and why the number of them is exactly nine.

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<sup>6</sup>These formulas work for  $n > 3$ . If  $n = 3$  we set

$$\begin{aligned}\tilde{C}_0^2 &= \det \left( (y_0, y_2)^t (y_{-2}, y_0) \right), \\ \tilde{C}_1^2 &= \det \left( (y_0, y_3)^t (y_0, y_3) \right) - \det \left( (y_0, y_2)^t (y_0, a_3 y_2 + b_2 y_3) \right) + \det \left( (Q(-c)y_0, y_3)^t (y_{-2}, y_0) \right)\end{aligned}$$

- Do there exist other Poisson structures compatible with the one in  $q_n(\tau)$  where  $n \geq 5$ ?
- There exist other elliptic Poisson algebras, for example  $q_{n,k}(\tau)$  where  $1 \leq k < n$  and  $n, k$  are coprime [5]. Functional constructions of these Poisson algebras can be found in [7]. Do there exist Poisson structures compatible with the one in  $q_{n,k}(\tau)$ ? Note that  $q_{n,1}(\tau) = q_n(\tau)$ ,  $q_{n,n-1}(\tau)$  is trivial so the first nontrivial example other than  $q_n(\tau)$  is  $q_{5,2}(\tau)$ .

We plan to address these problems elsewhere.

## Appendix: Elliptic and $\theta$ -functions

Fix  $\tau \in \mathbb{C}$  such that  $\operatorname{Im} \tau > 0$ . Let  $\Gamma = \{k + l\tau; k, l \in \mathbb{Z}\} \subset \mathbb{C}$  be an integral lattice generated by 1 and  $\tau$ . The Weierstrass zeta function is defined as follows:

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \in \Gamma \setminus \{0\}} \left( \frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right)$$

The function  $\zeta(z)$  is not elliptic but one has  $\zeta(z + \omega) = \zeta(z) + \eta(\omega)$  where  $\eta : \Gamma \rightarrow \mathbb{C}$  is a  $\mathbb{Z}$ -linear function. The functions  $\zeta(z_1 - z_2) - \zeta(z_1) + \zeta(z_2)$  and  $\zeta'(z)$  are elliptic. Moreover, a function  $c_1\zeta(z - u_1) + \dots + c_m\zeta(z - u_m)$  is elliptic in  $z$  if  $c_1 + \dots + c_m = 0$ .

Let  $n \in \mathbb{N}$ . We denote by  $\Theta_n(\tau)$  the space of the entire functions of one variable satisfying the following relations:

$$f(z + 1) = f(z), \quad f(z + \tau) = (-1)^n e^{-2\pi i n z} f(z)$$

It is known [8] that  $\dim \Theta_n(\tau) = n$ , every function  $f \in \Theta_n(\tau)$  has exactly  $n$  zeros modulo  $\Gamma$  (counted according to their multiplicities), and the sum of these zeros modulo  $\Gamma$  is equal to zero. Let  $\theta(z) = \sum_{\alpha \in \mathbb{Z}} (-1)^\alpha e^{2\pi i (\alpha z + \frac{\alpha(\alpha-1)}{2} \tau)}$ . It is clear that  $\theta(z) \in \Theta_n(\tau)$ . We have  $\theta(0) = 0$  and this is the only zero modulo  $\Gamma$ . Moreover, there exist functions  $\{\theta_\alpha(z); \alpha \in \mathbb{Z}/n\mathbb{Z}\} \subset \Theta_n(\tau)$ . These functions are uniquely defined (up to multiplication by a common constant) by the following identities:

$$\theta_\alpha \left( z + \frac{1}{n} \right) = e^{2\pi i \frac{\alpha}{n}} \theta_\alpha(z), \quad \theta_\alpha \left( z + \frac{1}{n} \tau \right) = -e^{-2\pi i (z + \frac{1}{n} - \frac{n-1}{2n} \tau)} \theta_\alpha(z)$$

and form a basis of the linear space  $\Theta_n(\tau)$ .

Note that  $\zeta(z)$  as well as  $\theta(z)$ ,  $\theta_\alpha(z)$  are functions in two variables:  $z \in \mathbb{C}$  and modular parameter  $\tau$ . Therefore, they can be written as  $\zeta(z, \tau)$  and  $\theta(z, \tau)$ ,  $\theta_\alpha(z, \tau)$ .

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